Infinite sum in the third degree.

https://www.linkedin.com/feed/update/urn:li:activity:6727361719741628416

Find the value of

$$\left(1+\frac{2}{6}+\frac{2\cdot 5}{6\cdot 12}+\frac{2\cdot 5\cdot 8}{6\cdot 12\cdot 18}+\ldots\right)^3$$

Solution by Arkady Alt, **San Jose** ,**California**, **USA**.

Since $(1-x)^{-2/3}=1+\sum_{n=1}^{\infty}(-x)^n\binom{-\frac{2}{3}}{n}, |x|<1$ then

$$1+\frac{2}{6}+\frac{2\cdot 5}{6\cdot 12}+\frac{2\cdot 5\cdot 8}{6\cdot 12\cdot 18}+\ldots=1+\sum_{n=1}^{\infty}\frac{\prod_{k=1}^{n}(3k-1)}{6^n n!}=$$

$$1+\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^n\prod_{k=1}^{n}\frac{1-3k}{3}\cdot\frac{1}{n!}=1+\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^n\frac{1}{n!}\prod_{k=1}^{n}\left(\frac{1}{3}-k\right)=$$

$$1+\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^n\frac{1}{n!}\prod_{k=1}^{n}\left(-\frac{2}{3}-(k-1)\right)=1+\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^n\binom{-\frac{2}{3}}{n}=$$

$$\left(1-\frac{1}{2}\right)^{-2/3}=2^{\frac{2}{3}}$$

and, therefore, . $\left(1+\frac{2}{6}+\frac{2\cdot 5}{6\cdot 12}+\frac{2\cdot 5\cdot 8}{6\cdot 12\cdot 18}+\ldots\right)^3=4.$